A Fusion Center Approach for Estimation Using Quantized Measurements

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***Abstract*—A fusion center approach to estimate a constant location parameter using quantized noisy measurements from multiple sensors is presented. The asymptotic estimation performance is obtained and simulations for different numbers of sensors under Gaussian and Cauchy noise are used for validation. A performance comparison under constrained communication bandwidth between a fusion center approach with two low resolution sensors and a high resolution single sensor approach is presented to motivate the use of low resolution sensor networks.**

***Index Terms*—Parameter estimation, quantization, adaptive algorithm, wireless sensor networks.**

I. INTRODUCTION

Technological advances in sensing and communication devices allowed wireless sensor networks to emerge as a new domain of research. This new domain has direct applications in military, health and commercial areas [1].

The deployment of a large quantity of sensors to track or estimate parameters of physical models creates a new spectrum of problems for the design of the system. Stringent constraints on costs and energy must now be considered. In a sensor network, these constraints can be viewed mainly as bandwidth and complexity constraints.

A direct way of dealing with these constraints is to allow the sensors to quantize their measurements. As in a sensing system the objective is to estimate underlying parameters embedded in noise, the quantization process must be designed to optimize estimation performance. Optimal estimation performance of a constant scalar parameter based on uniformly quantized noisy measurements was studied in [2]. It was shown that a good way of selecting the quantizer input offset was to use feedback information from the quantizer output. A fusion center approach was also proposed, where the fusion center receives the quantized information from all sensors and broadcast the last estimate to the sensors, which use it as the quantizer input offset. The use of estimates of the parameter as the input offset was motivated by the fact that the asymptotic optimal estimation variance, given by the Cramér–Rao bound (CRB), was observed to be minimized by placing the bias exactly at the threshold. In the binary measurement and Gaussian noise case, it was observed that by placing the threshold in this way, the CRB for quantized measurements were only $\frac{\pi}{2}$ times the CRB for continuous measurements. The small loss due to quantization lead many others [3]–[5] to study and develop algorithms for estimation using multiple sensors and binary quantizers.

In this paper a fusion center approach will also be developed. The estimator in the fusion center will be based on a low complexity recursive algorithm for which the asymptotic variance can be studied using stochastic approximation theory. To check for the validity of the results, the theoretical results will be compared to the simulation of the algorithm for different number of sensors with Gaussian and Cauchy distributed noise. At the end, a simulation comparing the algorithm performance with one sensor and five quantization bits and two sensors, one with two quantization bits and the other with three bits, will show the superiority of the low resolution multisensor approach when the total bandwidth is fixed to a constant number of bits.

II. PROBLEM STATEMENT AND ESTIMATOR

A constant scalar parameter x is measured in N sensors. Each sensor measures the parameter with additive noise

$$Y_k^{(j)} = x + V_k^{(j)}, \quad \text{for} \quad j \in \{1, \cdots, N\}, \tag{1}$$

where $V_k^{(j)}$ is the noise random variable (r.v.) for the sample k obtained at the sensor j. The noise is supposed to be independent between sensors and independent and identically distributed (i.i.d) with respect to (w.r.t.) the sample index k. The measurements at each sensor are quantized by scalar quantizers with adjustable input offsets and gains and then they are sent to a fusion center that will estimate the parameter.

The adjustable quantizers characteristics are their input gains $\frac{1}{\Delta_k^{(j)}}$, input offsets $b_k^{(j)}$ and the vectors of thresholds (considered to be static) that define the $N_I^{(j)}$ quantizer intervals

$$\boldsymbol{\tau}^{(j)} = \left[\tau^{(j)}_{-\frac{N_I}{2}} \ \cdots \ \tau^{(j)}_{-1} \ \tau^{(j)}_{0} \ \tau^{(j)}_{1} \ \cdots \ \tau^{(j)}_{\frac{N_I}{2}}\right].$$

The output of quantizer j is given by

$$i_k^{(j)} = Q^{(j)}\left(\frac{Y_k^{(j)} - b_k^{(j)}}{\Delta_k^{(j)}}\right) = i \operatorname{sign}\left(Y_k^{(j)} - b_k^{(j)}\right),$$
$$\text{for } \frac{\left|Y_k^{(j)} - b_k^{(j)}\right|}{\Delta_k^{(j)}} \in \left[\tau^{(j)}_{i-1}, \tau^{(j)}_{i}\right). \tag{2}$$

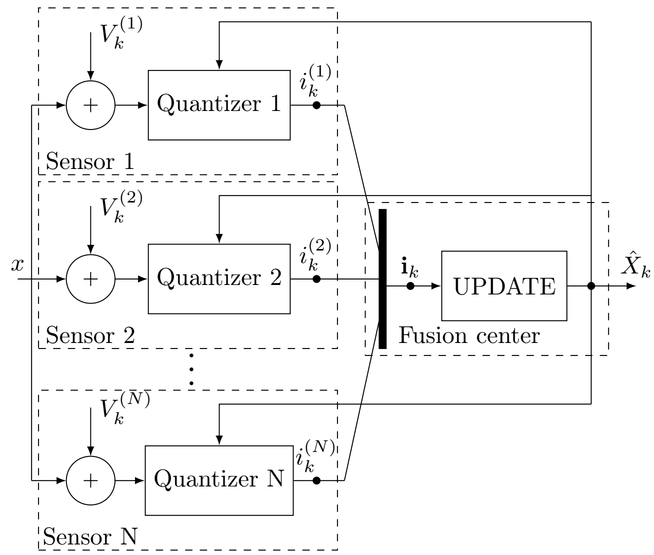

Fig. 1. Scheme representing the sensor network. The last estimate is broadcasted by the fusion center through a perfect channel to be used as quantizer offset.

The set of quantizer outputs of the sensor j is $\mathcal{I}_{(j)} = \left\{-\frac{N_I^{(j)}}{2}, \cdots, -1, 1, \cdots, \frac{N_I^{(j)}}{2}\right\}$.

The following assumptions on the quantizer and on the measurement noise will be considered:

- The marginal cumulative distribution function (CDF) for the noise r.v. will be denoted by $F^j(v)$. $V_k^{(j)}$ has a probability density function (PDF) $f^{(j)}(v)$ that is an even function.
- The quantizers have symmetric thresholds $\tau_i^{(j)} = -\tau_{-i}^{(j)}$ with $\tau_0^{(j)} = 0$ and $\tau_{\frac{N_I}{2}}^{(j)} = +\infty$.
- The fusion center will broadcast the last parameter estimate $\hat{X}_{k-1}$ without noise to the sensors that will use it as input offset:
$$b_k^{(j)} = \hat{X}_{k-1}. \tag{3}$$
- The noise CDFs have known scale parameters $\delta^{(j)}$, thus the CDFs can be written as $F^{(j)}(v) = F_n^{(j)}\left(\frac{v}{\delta^{(j)}}\right)$, where F_n is the CDF for $\delta^{(j)} = 1$. The quantizer gain can be used to normalize the input w.r.t. the noise scale parameter
$$\Delta_k^{(j)} = c_\delta^{(j)}\delta^{(j)}, \tag{4}$$
while the free constant parameter $c_\delta^{(j)}$ can be chosen to optimize estimation performance when the thresholds are fixed.

The last two assumptions are used to enhance the estimation performance by placing the dynamic range of the quantizer in a region with richer statistical information.

The general scheme is depicted in Fig. 1, where the UPDATE block contains an online estimator of the parameter. In a direct approach to solve this estimation problem an online version of the maximum likelihood estimator could be used. This would require unbounded memory capacity because all previous samples would have to be stored and also all previous estimators, as they are the previous central thresholds. To find the maximum likelihood estimate, a complex iterative optimization procedure based on gradients would be necessary. A solution to reduce complexity and memory requirements is to consider that the procedure iterates only one time per sample set and that it uses only the present set of measurements, this leads to the following algorithm:

$$\hat{X}_k = \hat{X}_{k-1} + \gamma_k \eta(\mathbf{i}_k), \tag{5}$$

where γ_k is a sequence of positive gains, $\mathbf{i}_k$ is the vector of quantized observations $\left[i_k^{(1)}, \cdots, i_k^{(N)}\right]^T$ and $\eta[\mathbf{i}]$ is a quantizer output coefficient, defined as a function from $\left\{\mathcal{I}_{(1)}, \cdots, \mathcal{I}_{(N)}\right\}$ to $\mathbb{R}$.

III. Estimation performance

The performance of the estimator (5) can be studied using the theory developed in [6] for adaptive algorithms.

It is shown in [6] that algorithms of the form (5) when used for estimating constant parameters must have decreasing gains as follows

$$\gamma_k = \frac{\gamma}{k}. \tag{6}$$

Also using the results from [6, Chap. 3], the asymptotic variance of the estimation error can be obtained under the condition that the mean error converges to zero as $k \longrightarrow \infty$. To prove this convergence, it would be sufficient to use an ordinary differential equation (ODE) approximation of (5) and then prove global convergence properties for the ODE using Lyapunov theory. In this work, such analysis will not be considered, only the mean behavior when $\hat{X}_k = x$ will be studied.

When $\hat{X}_k = x$, the mean increment $E\left(\hat{X}_k - \hat{X}_{k-1}\right)$ is given by

$$E\left(\hat{X}_k - \hat{X}_{k-1}\right) = \gamma_k \mathbb{E}[\eta(\mathbf{i})] = \gamma_k \boldsymbol{\eta}^T \mathbf{F}_d^{vec}, \tag{7}$$

where $\boldsymbol{\eta}$ is a vector regrouping all possible values of the output coefficients
$\left[\eta\left(i_{-\frac{N_I^{(1)}}{2}}, \cdots, i_{-\frac{N_I^{(N)}}{2}}\right) \cdots \eta\left(i_{\frac{N_I^{(1)}}{2}}, \cdots, i_{\frac{N_I^{(N)}}{2}}\right)\right]^T$
and $\mathbf{F}_d^{vec}$ is a vector defined in the same way but with elements given by

$$F_d(\mathbf{i}) = \prod_{j=1}^{N} F_d^{(j)}\left(i^{(j)}\right), \tag{8}$$

where $F_d^{(j)}\left(i^{(j)}\right)$ is the probability of having the output $i^{(j)}$ at the sensor j when $\hat{X}_k = x$:

$$F_d^{(j)}\left(i^{(j)}\right) = \begin{cases} F^{(j)}\left(\tau_i^{(j)}c_\delta^{(j)}\right) - F^{(j)}\left(\tau_{i-1}^{(j)}c_\delta^{(j)}\right) \\ \qquad \text{if } i^{(j)} \in \left\{1, \cdots, \frac{N_I^{(j)}}{2}\right\}, \\ F^{(j)}\left(\tau_{i+1}^{(j)}c_\delta^{(j)}\right) - F^{(j)}\left(\tau_i^{(j)}c_\delta^{(j)}\right) \\ \qquad \text{if } i^{(j)} \in \left\{-1, \cdots, -\frac{N_I^{(j)}}{2}\right\}. \end{cases} \tag{9}$$

Thus, the following condition is needed to have an equilibrium point at the true parameter:

$$\boldsymbol{\eta}^T \mathbf{F}_d^{vec} = 0, \qquad (10)$$

Note that this is a necessary condition for asymptotic unbiasedness of the algorithm.

Assuming that the algorithm is asymptotically unbiased, the results in [6, pp. 110-113] can be applied to obtain the asymptotic distribution of the estimation error, the optimal gain $\gamma = \gamma^\star$ and minimal estimation variance σ_ϵ^2. The asymptotic error is Gaussian distributed and it is given as follows

$$k^{\frac{1}{2}} \epsilon_k \underset{k\to\infty}{\rightsquigarrow} \mathcal{N}\left(0, \sigma_\epsilon^2\right) \qquad (11)$$

The optimal γ and minimal σ_ϵ^2 are then given by

$$\gamma^\star = \frac{1}{\boldsymbol{\eta}^T \mathbf{f}_d \mathbf{f}_d^T \boldsymbol{\eta}} \qquad (12)$$

and

$$\sigma_\epsilon^2 = \frac{\boldsymbol{\eta}^T \boldsymbol{F}_d \boldsymbol{\eta}}{\boldsymbol{\eta}^T \mathbf{f}_d \mathbf{f}_d^T}. \qquad (13)$$

The matrix $\boldsymbol{F}_d$ is a diagonal matrix diag $[\mathbf{F}_d^{vec}]$ and $\mathbf{f}_d$ is the vector form (as $\boldsymbol{\eta}$ and $\mathbf{F}_d^{vec}$) regrouping the elements

$$f_d\left(\mathbf{i}\right) = \sum_{j=1}^{N} f_d\left(i^{(j)}\right) \prod_{j'=1,\, j'\neq j}^{N} F_d^{(j')}\left(i^{(j')}\right), \qquad (14)$$

where

$$f_d^{(j)}\left(i^{(j)}\right) = \begin{cases} f^{(j)}\left(\tau_{i-1}^{(j)} c_\delta^{(j)}\right) - f^{(j)}\left(\tau_i^{(j)} c_\delta^{(j)}\right) \\ \qquad \text{if } i^{(j)} \in \left\{1, \cdots, \frac{N_I^{(j)}}{2}\right\}, \\ f^{(j)}\left(\tau_i^{(j)} c_\delta^{(j)}\right) - f^{(j)}\left(\tau_{i+1}^{(j)} c_\delta^{(j)}\right) \\ \qquad \text{if } i^{(j)} \in \left\{-1, \cdots, -\frac{N_I^{(j)}}{2}\right\}. \end{cases} \qquad (15)$$

The asymptotic performance can also be optimized through the choice of $\boldsymbol{\eta}$, this can be done by minimizing (13) w.r.t. $\boldsymbol{\eta}$ under the equilibrium constraint (10). This problem can be cast as a modified eigenvalue problem for which the solution is given in [7]. This leads to the following results:

$$\boldsymbol{\eta} = \mathbf{F}_d^{-1}\mathbf{f}_d - \mathbb{1}\mathbf{f}_d = \mathbf{F}_d^{-1}\mathbf{f}_d, \qquad (16)$$

where $\mathbb{1}$ is a square matrix of ones, the second equality comes from the symmetry assumptions and

$$\sigma_\epsilon^2 = \gamma^\star = \frac{1}{\mathbf{f}_d^T \mathbf{F}_d \mathbf{f}_d} \qquad (17)$$

Using the symmetry assumptions, it is possible to write the results above as the following sums:

$$\boldsymbol{\eta}\left(\mathbf{i}\right) = \sum_{j=1}^{N} \frac{f_d^{(j)}\left(i^{(j)}\right)}{F_d^{(j)}\left(i^{(j)}\right)} \qquad (18)$$

and

$$\sigma_\epsilon^2 = \gamma^\star = \frac{1}{\sum_{j=1}^{N} \sum_{i^{(j)}} \frac{f_d^{(j)\,2}\left(i^{(j)}\right)}{F_d^{(j)}\left(i^{(j)}\right)}} \qquad (19)$$

The output coefficients can be seen as a sum of the score functions for the quantized measurements of the different sensors when the central thresholds are placed exactly at the parameter and the asymptotic variance is the inverse of the sum of the Fisher information for the measurements from the sensors also when the central thresholds are placed at the parameter. This indicates that the estimator can be implemented using tables with $N_I^{(j)}$ coefficients only and also that the estimator is asymptotically efficient.

The last free parameters are $\boldsymbol{\tau}^{(j)}$ and $c_\delta^{(j)}$, both can be chosen to maximize the individual Fisher information for each sensor. The optimization through $\boldsymbol{\tau}^{(j)}$ is a difficult problem that will not be treated here. In the following sections, the quantizer will be supposed to be uniform with unitary quantization intervals and $c_\delta^{(j)}$ will be used for optimizing the estimation performance.

IV. Simulations

The validity of the results will now be verified through simulations. All the sensors within a simulation will be considered to have the same type of noise and the same noise scale factor δ. The noise considered will be Gaussian or Cauchy distributed with the following PDFs respectively:

$$\begin{aligned} f_G\left(v\right) &= \frac{1}{\delta\sqrt{\pi}} e^{-\left(\frac{v}{\delta}\right)^2}, \\ f_C\left(v\right) &= \frac{1}{\delta\pi\left(1 + \left(\frac{v}{\delta}\right)^2\right)}. \end{aligned} \qquad (20)$$

Optimization w.r.t. c_δ (the same gain for all sensors in this case, as the noise is identically distributed) will be done by searching the maximum of the corresponding Fisher information in a fine grid. After finding the optimal c_δ, the coefficients $\frac{f_d(i)}{F_d(i)}$ and the gain $\gamma^\star$ can be calculated.

For all the following simulations, the length of the block of samples will be 5000 and for evaluating the mean squared error (MSE) the average of the squared error will be calculated using 50000 blocks, the noise scale will be $\delta = 1$ for both types of noise. The parameter value and initial estimator value are $x = 0$ and $\hat{X}_0 = 1$.

In the first simulation, it will be considered that all the quantizers have $N_I = 4$ and N will be 1, 2 or 3, the results can be observed in Fig. 2 in log scale both in time and MSE. The simulated results are compared with the theoretical approximations, for this algorithm they are asymptotically equal to the optimal CRB. As it was expected the MSE decreases with the number of sensors and the simulated results are very close to the theoretical approximation for a large number of samples. To have a more appropriate comparison between different number of sensors, channel bandwidth constraints must be considered.

In the second simulation, the total bandwidth considered will be 5 bits. Two possible settings will be considered, a single sensor approach using the 5 bits ($N_I = 32$) and a multi-sensor approach with one sensor quantizing the measurements with 2 ($N_I = 4$) bits and the other with 3 bits ($N_I = 8$). The

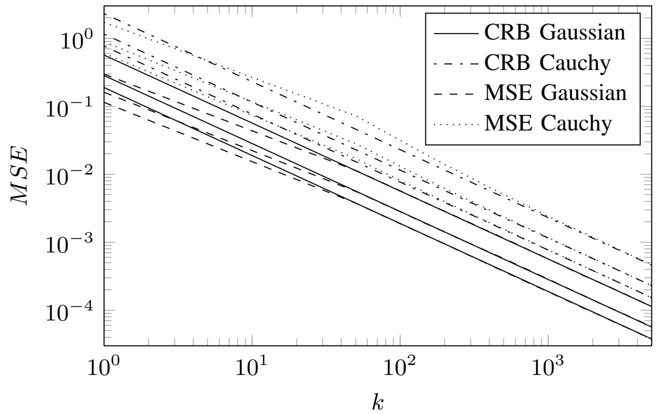

Fig. 2. Cramér–Rao bound and simulated mean squared error (MSE) for the adaptive algorithm when $N_B = 2$, $N = 1, 2, 3$ and the noise is Gaussian or Cauchy distributed. The curves are plotted in $\log\log$ scales for better visualization. In each set of curves the results for the three different number of sensors are represented, the highest MSE curves represents the performance for $N = 1$ and the lowest MSE represent $N = 3$.

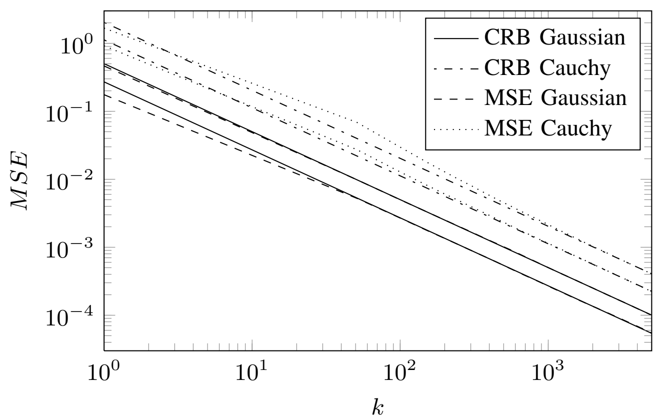

Fig. 3. Cramér–Rao bound and simulated mean squared error (MSE) for the adaptive algorithm for $N = 1$ and $N_B = 5$ and for $N = 2$, one sensor with $N_{B1} = 2$ and the other with $N_{B2} = 3$. For each set of results the higher curve represent the performance for $N = 1$.

results are shown in Fig. 3, also with a comparison with the CRBs. For both types of noise, the theoretical and simulated results show that the multisensor approach is superior.

V. Conclusions

In this work, a fusion center approach for estimation based on quantized measurements was presented. An online low complexity estimator was proposed to estimate a constant using quantized noisy measurements obtained from multiple sensors, the quantizers input offsets were considered to be equal to the last estimate broadcasted by the estimator. The estimation performance was studied and optimized. It was shown that the asymptotic error variance is equal to the CRB obtained when considering that the measurements are quantized with quantizers centered at the true parameter.

The theoretical results were validated through simulation under two types of measurement noise distribution, Gaussian and Cauchy, and different number of sensors. In a fixed total bandwidth context, it was observed that an approach with multiple sensors and low resolution quantizers was superior to a single sensor approach. Such observation motivates the use of low resolution sensor networks for estimation purposes.

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